Further Results on the Detectability of Known Signals in Gaussian Noise

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The detection of a completely known signal which may or may not be present in a finite sample of gaussian noise is considered from two points of view. The first examines the performance of a maximum likelihood detector operating on a finite set of discrete measurements of the stimulus as the set becomes large. The stimulus is either signal plus noise or noise alone. Examples are presented for signals in bandlimited noise, using as measurements either equispaced amplitude samples or derivatives at one instant in time. For both, the detectability grows without bound as the number of measurements is increased. The second point of view bases detection on a continuous measurement (linear integral operator) which maximizes the detectability. Solutions have been obtained when the noise has a rational power spectral density. The detector utilizes a cross-correlation between stimulus and signal which is well known and a mechanism, designated extrapolation detection, which involves evaluation of derivatives of the stimulus. The contribution of the derivative measurements to the detectability is examined as the noise approaches bandlimited noise and is found in many cases to grow without bound.

I. INTRODUCTION

The problem under consideration here is the detection of a completely known signal which may or may not be present in a finite sample of gaussian noise. That is, we imagine a situation similar to Fig. 1 in which a stimulus is made up of either signal plus noise or noise alone and we ask, given T seconds of this stimulus, how accurately can we decide whether or not the signal is present. The noise is thought of as having been produced by a stochastic process and thus the question is really one of statistical hypothesis testing.

This particular problem has been treated rather extensively,[1-7] and certain questions, even controversies, have arisen. These concern what

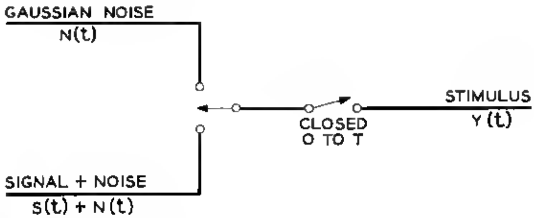

Fig. 1 — Diagram of problem under consideration.

constitutes a proper description for the stimulus, under what circumstances can the stimulus be characterized by a finite number of samples, and under what conditions is perfect detectability obtained, i.e., when is it always possible to detect the presence or absence of the signal. Peterson, Birdsall and Fox[6] have described the stimulus as being *Fourier series bandlimited* and by so doing have obtained quite different results from the other authors, who for the most part consider stationary gaussian noise. In many cases, finite-duration stimuli have been characterized by a finite number of samples usually chosen so they are independent, and maximum likelihood detectors operating on these samples have been developed. This has led to the equivalent of a correlation detection process in which the test statistic is the integral of the product of the stimulus and a function derived from the signal. Such detectors always produce finite detectability. On the other hand, Slepian[7] has pointed out by an argument involving analytic continuation that many signals can be perfectly separated from noise provided the noise is considered to have a bandlimited spectrum. Clearly some mechanism in addition to correlation detection is inherent in Slepian's result, and indeed he points out one such detector.

The results of Peterson, Birdsall and Fox have been used extensively for comparison with the performance achieved by humans and other animals, and questions as to the validity of such comparisons originally motivated this investigation. However, it seems very doubtful if the mechanisms which will be developed can have anything to do with perception. In addition, we have chosen to work with stationary gaussian noise rather than Fourier series bandlimited noise, the former being a much more satisfactory characterization of real noise.

Two different attempts to better understand the questions cited above have been undertaken. The first examines the performance of a maximum likelihood detector operating on a finite set of discrete measurements of the stimulus as the set becomes very large. The results show cases where the detectability grows without bound. Thus, the charac-

terization of the stimulus by a finite set of measurements is incomplete. However, in some cases, a law of diminisbing returns operates so that the rate of increase in detectability slows as the number of samples is increased.

The second study bases detection on a continuous measurement (linear integral operator), which is the solution of an optimizing integral equation. The test statistic so obtained has two parts, one similar to correlation detection, the other based on measurements of tbe derivatives of tbe stimulus. The contribution of this latter term is usually the smaller of the two, but, where the noise spectrum approaches a band-limited form, it may grow without bound. In addition, it may be important if the stimulus is very short.

Both maximum likelihood detection with a finite number of samples and the integral equation for the continuous statistic bave been previously presented. The new contributions arise from the more complete solutions which bave been obtained. The most significant result is undoubtedly the solution of the integral equation in closed form so that its characteristics and particularly its asymptotic properties for many-pole noise can be seen. The derivative detector, which will be termed *extrapolation detection*, was apparent from this solution.

II. DETECTION WITH A FINITE NUMBER OF SAMPLES

In this section we will derive the maximum likelihood detector for detecting a known signal in gaussian noise from a finite number of samples of the stimulus and apply this detector to two specific problems involving bandlimited noise. Each sample results from some linear operation on the stimulus and the samples need not be independent. The derivation of the detection equation differs only slightly from previously published work,[5] and is included to lead clearly into the specific problems, which are the principal new results. In the problems the behavior of the detector is studied as the number of samples becomes large, first when the samples consist simply of amplitude measurements of the stimulus and second when the samples are a set of derivatives at one point in time.

2.1 *Maximum Likelihood Detector*

The stimulus

$$Y(t) = \begin{cases} N(t) \\ N(t) + S(t) \end{cases} \qquad 0 \leqq t \leqq T, \tag{1}$$

is either a gaussian noise $N(t)$ or that noise plus a known signal $S(t)$ and is observed for the interval $0 \leq t \leq T$. The n samples,

$$Y_1, Y_2, \cdots, Y_n,$$

on which the detection is made are obtained by n linear operations $L_1, L_2, \cdots, L_n$ on the stimulus

$$Y_i = L_i[Y(t)] \qquad i = 1, \cdots, n.$$

Because of their linearity,

$$L_i[N(t) + S(t)] = L_i[N(t)] + L_i[S(t)] \equiv N_i + S_i,$$

and N_i will be gaussian random variables which may be completely characterized by their matrix β of correlation coefficients,

$$\beta_{ij} = E\langle N_i N_j \rangle,$$

and by their means which for simplicity will be assumed to be zero,

$$E\langle N_i \rangle = 0.$$

The density function of the Y_i samples when the stimulus is noise alone may then be written

$$f_N(y_1, \cdots, y_n) = (2\pi)^{-n/2} \mid \beta \mid^{-\frac{1}{2}} \exp \{-\tfrac{1}{2} \sum_{i,j} \beta_{ij}^{-1} y_i y_j\},$$

where $y_1, \cdots, y_n$ are the dummy arguments of the density function corresponding to $Y_1, \cdots, Y_n$ and $\mid \beta \mid$ is the determinant of β, with all sums going over the range 1 to n unless especially indicated otherwise. The density function of Y_i for signal plus noise is simply

$$f_{SN}(y_1, \cdots, y_n) = f_N(y_1 - S_1, \cdots, y_n - S_n)$$

because the signal is additive. Thus the likelihood ratio $L(y_1, \cdots, y_n)$ is

$$L(y_1, \cdots, y_n) = \frac{f_{SN}(y_1, \cdots, y_n)}{f_N(y_1, \cdots, y_n)},$$

which when evaluated for these density functions becomes

$$L(y_1, \cdots, y_n) = \exp \{-\tfrac{1}{2} \sum_{i,j} \beta_{ij}^{-1} S_i S_j\} \exp \{\sum_{i,j} \beta_{ij}^{-1} S_i y_j\}.$$

A maximum likelihood detector says that signal is present if test statistic $L(Y_1, \cdots, Y_n)$ is greater than some threshold α and will maximize the conditional probability of detecting a signal when it is present for a given conditional probability of indicating signal for noise alone.

However, L is a monotonic function of the statistic φ,

$$\varphi = \sum_{i,j} \beta_{ij}^{-1} S_i Y_j \,, \tag{2}$$

and consequently an equally good test is $\varphi > \alpha_c$, where α_c is an equivalent threshold. φ may be characterized by two density functions, one if the stimulus is noise alone, the other for signal plus noise. For noise alone, φ_N (the subscript "N" designates noise alone, "SN" signal plus noise) is gaussian with zero mean and variance

$$E\langle \varphi_N^2 \rangle = \sum_{i,j} \beta_{ij}^{-1} S_i S_j \,.$$

For signal plus noise φ_{SN} is also gaussian with the same variance but with mean

$$E\langle \varphi_{SN} \rangle = \sum_{i,j} \beta_{ij}^{-1} S_i S_j \,.$$

The density functions for φ are pictured on Fig. 2. The effectiveness of this detector as indicated by the signal detection probability at a given false alarm rate can be characterized by a single number d, which is the ratio of the squared mean of the signal plus noise distribution to the variance of either distribution. The larger d is, the more completely separated are the distributions on Fig. 2 and the higher will be the detection probability. This number d is then

$$d = \sum_{i,j} \beta_{ij}^{-1} S_i S_j \,. \tag{3}$$

An alternate form for the statistic φ from that given in (2) is

$$\varphi = \sum_{j} Z_j Y_j \,, \tag{4}$$

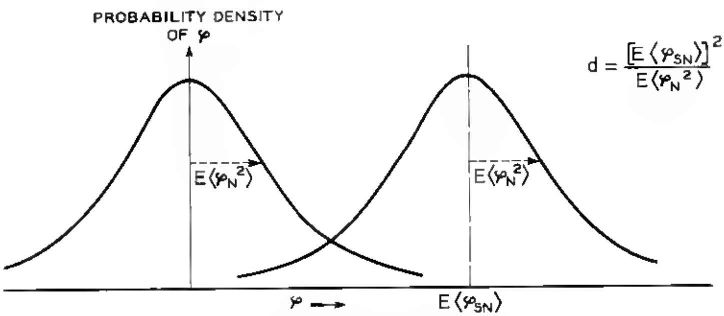

Fig. 2 — Two density functions characterizing statistic φ.

where the Z_j's are solutions to the equations

$$\sum_i \beta_{ij} Z_i = S_j \qquad j = 1,\cdots,n \tag{5}$$

and d may be expressed

$$d = \sum_j Z_j S_j . \tag{6}$$

This form is usually preferable for computations since it involves the solution of n linear equations rather than the inversion of an $n \times n$ matrix. In addition this form more closely resembles the integrals which will appear when continuous statistics are considered.

To summarize, a statistic φ which operates on a set of n correlated samples and which is equivalent to a maximum likelihood statistic has been developed. Signal is indicated if φ is greater than some threshold. φ is formed as a linear sum of the samples, it has a gaussian distribution, and it has the same variance for both noise alone and signal plus noise cases. The performance of the detector may be characterized by a single number $d = [E\langle\varphi_{SN}\rangle]^2/E\langle\varphi_N^{\,2}\rangle$, the larger the d, the better the performance.

2.2 *Detection of Sinusoid in Bandlimited Noise with Time Samples*

The argument presented by Slepian[7] indicates that theoretically, because of the analytic nature of the noise, a sinusoid can always be detected in spectral bandlimited noise. However, this result says nothing about how fast the detectability increases with the complexity of the detector. In this section an example is examined in which the stimulus is time sampled with n samples equally spaced over the interval $0 \leqq t \leqq T$ and detectability is computed as a function of n. In addition to the general behavior of this function, it is of special interest to note whether any peculiarities occur at $n = 2WT$ (the Nyquist rate), W being the noise bandwidth, since this is the maximum number of independent samples which may be formed. The correlation function of the noise is

$$R(\tau) = E\langle N(t)N(t+\tau)\rangle = \frac{\sin 2\pi W\tau}{2\pi W\tau},$$

where the noise has unit mean square amplitude so the matrix of correlation coefficients β_{ij} can be written

$$\beta_{ij} = \frac{\sin \pi \dfrac{n_q}{n}(i-j)}{\pi \dfrac{n_q}{n}(i-j)}$$

with

$$n_q = 2WT.$$

Unfortunately, no analytic way for either inverting this matrix or solving (5) is known, hence the detectability was computed numerically. This computation was carried out on an IBM 704 machine for a signal with frequency centered in the noise band

$$S_i = A \sin \frac{\pi}{2}\left(\frac{n_q}{n} i - \frac{n_q}{2n}\right),$$

A being the amplitude and $\pi n_q/4n$ being an arbitrary phase chosen for computational convenience. The normalized results of a solution of (5) and (6) are presented on Fig. 3, where d/A^2 is given as a function of the number of samples n/n_q and of the stimulus duration in terms of the number of independent samples n_q. The curves exhibit a knee, not at $n = n_q$ but for n a bit larger than n_q. Detectability continues to increase but the rate of increase becomes imperceptible. The curves are all carried out to a matrix of size 128 × 128, which is the limit of the capacity of the computer program. Double precision aritbmetic and a sufficient error analysis were used to insure the accuracy of the results. The increase in detectability beyond $n = n_q$ is essentially equivalent to that which would be obtained by increasing T to $T + 2/W$ and sampling at the Nyquist rate. Heuristically we can say that, by adding extra points inside the interval, it is quite easy to predict $N(t)$ two independent sample times beyond each end of tbe interval, but very hard to predict further. In an unpublished proof Slepian has shown that the quadratic form for d given by (3) does become infinite for bandlimited noise as n becomes infinite. However, the present example indicates it increases at an exceedingly slow rate. Clearly a statistic which improves more rapidly is desirable, and such is evaluated in the next section.

2.3 *Detection of a Constant in Bandlimited Noise Using Derivatives*

The solution for the optimum integral operator detector carried out in the next section produced a statistic involving derivatives of the stimulus. This result suggests trying derivatives for bandlimited noise, particularly since all derivatives of a bandlimited stimulus exist. Consequently, the detectability achieved by n samples, which are the stimulus and its $n - 1$ derivatives evaluated at one point in time, is studied. This quantity, as will be seen, has the pleasant characteristics of being analytically rather than only numerically determinable and of increasing uniformly with n rather than exhibiting the knee curves of the time samples. A curious property is that the duration of the stimulus is no

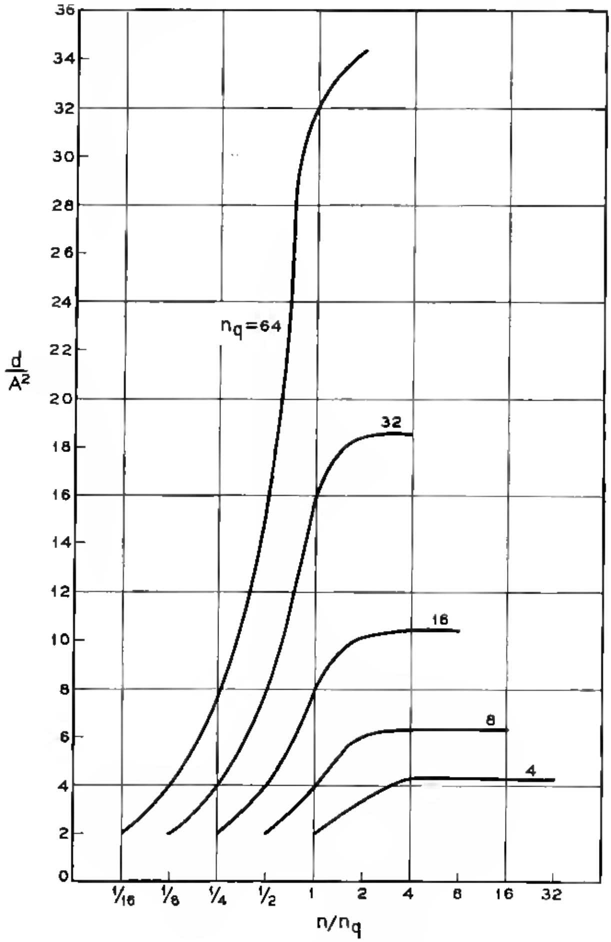

Fig. 3 — Normalized results of a solution of (5) and (6), with d/A^2 as a function of number of samples n/n_q and of stimulus duration in terms of number of independent samples n_q .

longer a factor in detectability since, theoretically at least, any number of derivatives can be measured from as short a sample as desired.

Detectability can again be computed from (5) and (6), where

$$\beta_{rs} = E\langle N^{(r-1)}(0)N^{(s-1)}(0)\rangle$$

is the correlation of the $r - 1$ and $s - 1$ derivatives,

$$N^{(s)}(t) \equiv \frac{d^{s-1}N(t)}{dt^{s-1}}.$$

The correlation coefficient may be written

$$\beta_{rs} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(\omega)(-j\omega)^{r-1}(j\omega)^{s-1}\, d\omega, \tag{7}$$

where $G(\omega)$ is the power spectrum of the noise

$$G(\omega) = \int_{-\infty}^{+\infty} E\langle N(t)N(t+\tau)\rangle e^{-j\omega\tau}\, d\tau.$$

If bandlimited noise with a flat spectrum from -1 to $+1$ rad/second and unit rms amplitude is selected, then (7) yields

$$\beta_{rs} = \begin{cases} \left(\dfrac{1}{r+s-1}\right)(-1)^{\frac{3}{2}r+\frac{1}{2}s} & \text{if } r+s \text{ is even} \\[2ex] 0 & \text{if } r+s \text{ is odd.} \end{cases}$$

A solution for (5) and (6) with these coefficients can be effected, since the determinants involved are reducible to a form with a solution attributed to Cauchy. The answer can probably be written on a large enough sheet of paper for signals having simple derivatives such as sinusoids, but the result is especially compact for a constant for which

$$S(0) = K, \quad S^{(n)}(0) \equiv \frac{d^{n-1}S(t)}{dt^{n-1}} = 0 \qquad n = 2,3,\cdots.$$

The evaluation, carried out in Appendix A, yields for d

$$d = K^2 \left[\frac{(2m)!}{2^{2m-1}m!(m-1)!}\right]^2, \tag{8}$$

where

$$m = \begin{cases} \dfrac{n}{2} & \text{for } n \text{ even} \\[2ex] \dfrac{n}{2} + \dfrac{1}{2} & \text{for } n \text{ odd.} \end{cases}$$

The asymptotic behavior of d for large m can be seen by substituting Stirling's approximation

$$a! \approx \sqrt{2\pi}\, \exp\{-a + (\log a)(a + \tfrac{1}{2})\}$$

for the factorials in (8), thus reducing it to

$$d \approx \frac{4K^2}{\pi} e^{1/(6m^2)} m. \tag{9}$$

The approximation is within 2 per cent for $m \geqq 20$.

Equations (8) and (9) exhibit the behavior of a statistic in which d increases linearly with the number of samples, each sample being a derivative. A similar behavior will be shown for rational noise where one term in the detectability depends linearly on the number of derivatives which exist and form part of the statistic. The bandlimited noise differs from the rational noise in that all its derivatives theoretically exist and the detectability can be made, at least theoretically, as good as desired by making m large enough. Obviously, in any practical case, the number of derivatives which can be estimated is limited. In addition, the characterization of the random process as gaussian undoubtedly fails for high enough derivatives.

Equations (8) and (9) are derived only for a signal which is a constant. However, a similar dependence on m would probably occur for sinusoidal signals.

The prominence of derivatives as an effective statistic for both bandlimited and rational noise gives a possible indication why detectability based on equally spaced time samples increases so slowly. These, being uniformly distributed, give poor estimates of derivatives. A more effective distribution might well be n_q independent samples spaced uniformly over the interval and the rest of the samples clustered as closely as possible about two points at each end of the interval. Such arrangement is suggested by statistics for the rational noise case.

III. DETECTION WITH CONTINUOUS SAMPLING

The preceding section discussed the detection of a known signal in bandlimited noise using a finite number of samples of the stimulus as a statistic. In this section we consider the detection of a known signal in gaussian noise using as the statistic a continuous measure of the stimulus over an interval T in length. The noise is now taken to have a rational power spectral density; that is, its power spectrum can be represented at the ratio of two polynomials in ω^2. Such noise can be thought of as resulting from the passage of ideal white gaussian noise through a finite linear lumped-element filter, although it need not actually have been produced in this way. For the purposes of the analysis, it is convenient to think of the situation as shown in Fig. 4. White gaussian noise is passed through a filter whose transfer function is $H(s)$, (Laplace transform of its impulse response) and to this may or may not be added the known signal $S(t)$. T seconds of the combination form the stimulus $Y(t)$. The problem is to decide from an examination of the stimulus whether or not the signal was present.

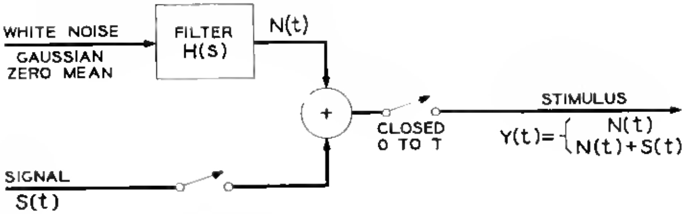

Fig. 4 — Diagram of continuous sampling situation.

The detection scheme in this case is essentially an extension of the finite sampling procedure. One asks for that linear integral operator which will extract from the stimulus a statistic giving the maximum detectability. Thus, the statistic is obtained from

$$\varphi = \int_0^T Y(t)Z(t)\,dt, \tag{10}$$

where $Z(t)$ is that function of time which maximizes the detectability. Because the noise is gaussian of zero mean and the signal (when present) is simply added to the noise, the statistic φ again has a gaussian probability density function whose mean value is zero or not zero according to the absence or presence of the signal and whose variance is the same with or without the signal. Thus it is reasonable to again define the detectability measure d as

$$d = \frac{[E(\varphi_{SN})]^2}{E(\varphi_N^2)}. \tag{11}$$

The optimization problem is thus to find $Z(t)$ which maximizes d or, that which is equivalent, to find $Z(t)$ which minimizes $E(\varphi_N^2)$ while holding $E(\varphi_{SN})$ constant. This latter form is a straightforward calculus of variation problem and its solution, the details of which are omitted, leads to the following integral equation for $Z(t)$:

$$\int_0^T R(t-u)Z(u)\,du = S(t) \qquad 0 \leq t \leq T, \tag{12}$$

where $R(\tau)$ is the autocorrelation function of the noise,

$$R(\tau) = E[N(t)N(t+\tau)].$$

When (12) is satisfied, the detectability can be written

$$d = \int_0^T Z(t)S(t)\,dt. \tag{13}$$

The discussion up to this point has not required that the noise have a rational spectral density. Unfortunately, it does not appear possible to carry (13) any further without actually solving (12) for $Z(t)$, and this has only been done in certain special cases. In particular, if the noise spectral density is the reciprocal of a polynomial, the solution for (12) can be exhibited in some detail; and furthermore if the signal is a sine wave, an exponential, or a constant the detectability can be expressed in a surprisingly simple form.

3.1 *All-Pole Noise*

If the noise has a spectral density $G(\omega)$,

$$G(\omega) = \int_{-\infty}^{+\infty} R(\tau)e^{-j\omega\tau}\, d\tau,$$

which is rational and contains only poles ($2N$ in number), it can be written in the form

$$G(\omega) = \frac{1}{a_0 - a_2\omega^2 + a_4\omega^4 - \cdots \pm a_{2N}\omega^{2N}}. \tag{14}$$

Such a noise could have been produced by passing white noise of unit spectral density through a filter whose transfer function $H(s)$ has N poles,

$$H(s) = \frac{1}{b_0 + b_1 s + b_2 s^2 + \cdots + b_N s^N} = \frac{1}{P(s)}, \tag{15}$$

and the poles can be placed in evidence by writing the denominator polynomial $P(s)$ as

$$P(s) = \sum_{k=0}^{N} b_k s^k = b_N(s - \gamma_1)(s - \gamma_2),\cdots,(s - \gamma_N), \tag{16}$$

where the γ's are (possibly) complex numbers giving the pole locations and each has a negative real part. In terms of $H(s)$, the spectral density can be written

$$G(\omega) = |\, H(j\omega)\,|^2.$$

Thus the noise can be described in a variety of ways—by the constants $a_0, a_2, \cdots, a_{2N}$, or the set $b_0, b_1, \cdots, b_N$, or the pole locations $\gamma_1, \gamma_2, \cdots, \gamma_N$ and one constant b_N, or even the magnitude and phase of the transfer function $H(s)$ for real frequencies. The particular set of parameters to be used will be chosen to simplify the final answer.

One characteristic of N-pole noise is that its first $N - 1$ derivatives exist, while the Nth and higher do not. Because of this it is clear that a necessary condition for finite detectability of a signal $S(t)$ is that its first $N - 1$ derivatives be continuous in the interval 0 to T. If this condition is not satisfied; that is, if among the $N - 1$ derivatives of $S(t)$ a discontinuity occurs, then the detectability is infinite. This is clearly true, because one could simply differentiate the stimulus enough times to produce a step function in the interval and this could always be found by measuring the change in the differentiated stimulus just before and just after the time of the step.

Using this N-pole noise, it is possible to exhibit explicit solutions to (12) and (13). Unfortunately, strictly speaking, (12) does not have a solution unless $S(t)$ and its derivatives up to order $N - 1$ satisfy a certain set of boundary conditions (boundaries at 0 and T). If $S(t)$ does not satisfy this set of boundary conditions, and in general for an arbitrary signal it will not, then (12) has a formal solution if $Z(t)$ includes delta functions and their derivatives to order $N - 1$ at the end points of the interval (approached from inside the interval). The details of this argument are presented in Appendix B, where it is shown that the solution to (12) is

$$\begin{aligned} Z(t) &= Z_c(t) + \sum_{i=0}^{N-1} [\alpha_i \delta^{(i)}(t) + \beta_i \delta^{(i)}(t - T)], \\ Z_c(t) &= \sum_{k=0}^{N} a_{2k} S^{(2k)}(t), \end{aligned} \tag{17}$$

where the superscript (n) indicates n-fold differentiation with respect to time, and the α's and β's are given by

$$\begin{aligned} \alpha_i &= \sum_{k=i}^{N-1} b_{k+1} U_2^{(k-i)}(0) \\ \beta_i &= \sum_{k=i}^{N-1} (-1)^k b_{k+1} U_1^{(k-i)}(T), \end{aligned} \qquad i = 0,1,2,\cdots,N-1 \tag{18}$$

with

$$U_1(t) = \sum_{k=0}^{N} b_k S^{(k)}(t) \qquad \text{and} \qquad U_2(t) = \sum_{k=0}^{N} (-1)^k b_k S^{(k)}(t).$$

When this $Z(t)$ is substituted in (13), the detectability becomes

$$d = \int_0^T Z_c(t) S(t)\, dt + \sum_{i=0}^{N-1} (-1)^i [\alpha_i S^{(i)}(0) + \beta_i S^{(i)}(T)]. \tag{19}$$

Among the several other ways of writing d, one which is convenient is the following (partly operator notation):

$$d = \int_0^T U_1^2(t)\,dt + \sum_{i=0}^{N-1} S^{(i)}(0)\{[(-1)^i P_i(p)P(-p) + P_i(-p)P(p)]S(t)\}_{t=0}, \tag{20}$$

where

$$P_i(x) = \sum_{k=i}^{N-1} b_{k+1}x^{k-i}$$

and p is the derivative operator d/dt. The derivatives of $S(t)$ and $U(t)$ at 0 and T are to be interpreted as the limit of the value of the derivatives approached from inside the interval.

The form of $Z(t)$ in (17) is quite interesting. The first part contributes a function of time which is similar to the conventional cross-correlation result. One simply multiplies the stimulus by this function and integrates the product. In the second part, the delta functions, when used with (10) to form the statistic, represent evaluating the stimulus and its first $N - 1$ derivatives at the ends of the interval. The derivatives at the ends give information about the stimulus outside the interval. Essentially they allow prediction or estimation of the stimulus outside the interval, and this information is to be added to that from straight cross-correlation. As N becomes larger the noise spectrum drops off faster at high frequencies and more derivatives of the stimulus are used (more derivatives of the noise exist); effectively, the stimulus can be predicted further outside the interval. Usually, this will mean that the signal can be detected better (see examples below).

3.2 *Damped Sinusoidal Signal*

As a particular example, consider the case in which the signal is a damped sine wave of arbitrary phase,

$$S(t) = Ae^{-\alpha t}\sin(\omega t + \Phi) = \bar{A}e^{\lambda t} + \bar{A}^*e^{\lambda^* t}, \tag{21}$$

where

$$\bar{A} = \frac{A}{2j}e^{j\Phi} \qquad \text{and} \qquad \lambda = -\alpha + j\omega.$$

Since the detectability is of primary interest, specific values for the co-

efficients of the delta functions will not be calculated. The details of the calculations are carried out in Appendix C, where it is shown that

$$d = 2\,\mathrm{Re}\left[\bar{A}^2\,\frac{P^2(\lambda)e^{2\lambda T} - P^2(-\lambda)}{2\lambda}\right] + 2\,|\,\bar{A}\,|^2\left[\frac{|\,P(\lambda)\,|^2\,e^{(\lambda+\lambda^*)T} - |\,P(-\lambda)\,|^2}{\lambda + \lambda^*}\right]. \tag{22}$$

3.3 *Exponential Signal*

For an exponential signal,

$$S(t) = Ae^{-\alpha t}$$

and the detectability from (22) becomes

$$d = \frac{A^2}{2\alpha}\,[P^2(\alpha) - P^2(-\alpha)\,e^{-2\alpha T}]. \tag{23}$$

With given signal parameters and noise filter, specific values of detectability can be calculated from this expression.

As the number of poles in the noise filter increases, $P^2(-\alpha)/P^2(\alpha) \to 0$, assuming the poles are bounded away from the imaginary axis and that $\alpha > 0$. In this case d becomes

$$d \to A^2P^2(\alpha)/2\alpha.$$

If as the number of poles is increased the DC gain of the filter is kept constant (or allowed to increase), then $P^2(\alpha)$ increases without bound. This can be seen by thinking of $P(\alpha)$ in factored form, which for constant DC gain looks like

$$P(\alpha) = b_0 \prod_{i=1}^{N} \frac{\alpha - \gamma_i}{-\gamma_i},$$

and noting that $|\,(\alpha - \gamma_i)/\gamma_i\,| > 1$. Thus, for fixed signal, more poles mean more detectability. A similar result obtains if $\alpha < 0$.

A noise filter of particular interest is a Butterworth filter, that is, one whose poles are uniformly distributed on a semicircle in the left-half plane. Such a filter gives noise whose spectrum is maximally flat lowpass and approaches ideal bandlimited noise as the number of poles increases. In this case, the approximate behavior of d for large N can

be calculated by taking the poles as smeared out on a semicircle of radius ω_0. Thus,

$$\ln \frac{P^2(\alpha)}{b_0^2} \cong \frac{2N}{\pi} \int_0^{\pi/2} \ln \left[1 + \left(\frac{\alpha}{\omega_0} \right)^2 + 2 \frac{\alpha}{\omega_0} \cos \Phi \right] d\Phi$$

and, therefore,

$$d \cong \frac{A^2}{2\alpha G(0)} B^N, \tag{24}$$

where

$$B = \exp \left\{ \frac{2}{\pi} \int_0^{\pi/2} \ln \left[1 + \left(\frac{\alpha}{\omega_0} \right)^2 + 2 \frac{\alpha}{\omega_0} \cos \Phi \right] d\Phi \right\}.$$

A sketch of B versus α/ω_0 is shown in Fig. 5. Clearly B is greater than one and the detectability grows exponentially for large N.

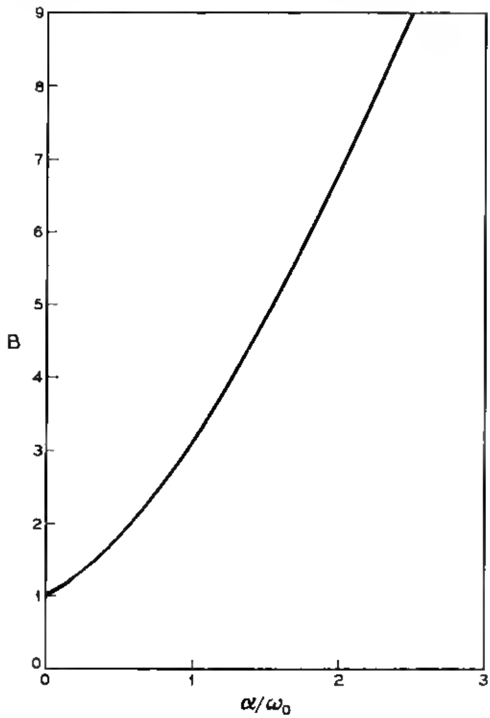

Fig. 5 — B vs. α/ω_0.

3.4 *Sinusoidal Signal*

For an undamped sine wave ($\alpha = 0$), (22) can be put in a more convenient form by using the magnitude and phase of the noise filter transfer function, $H(s)$, which can be written

$$H(j\omega) = \sqrt{G(\omega)}e^{-j\theta(\omega)}.$$

The angle $\theta(\omega)$ then is the phase lag of the noise filter, a function of frequency. In these terms (22) becomes

$$d = \frac{A^2}{2G(\omega)}\left[T + 2\dot{\theta}(\omega) - \frac{\sin(2\omega T + \theta + \Phi) + \sin 2(\theta - \Phi)}{2\omega}\right], \quad (25)$$

where

$$\dot{\theta} = d\theta/d\omega.$$

If $\omega T \gg 1$, that is, if the time is long so that there are many cycles of the sine wave in the interval, then the last term in (25) can be neglected. In conventional circuit analysis, $\dot{\theta}$ is generally considered the time delay of a network; thus, the detectability includes a term proportional to twice the time delay of the noise filter. Roughly, this says that the derivatives at the ends of the interval allow extension of the stimulus a distance equal to the time delay outside each end.

It is clear that the $\dot{\theta}$ term grows without bound as the number of poles bounded away from the imaginary axis is increased. In the particular case of noise with a maximally flat spectrum [Butterworth $H(s)$], this growth can be shown more explicitly. The contribution to $\dot{\theta}$ from a single pair of poles located at $-\omega_0 e^{\pm j\beta}$ is

$$\frac{2}{\omega_0}\frac{(\lambda^2 + 1)\cos\beta}{\lambda^4 + 1 + 2\lambda^2\cos 2\beta}, \qquad \lambda = \frac{\omega}{\omega_0}.$$

To add up the contributions from N poles on a semicircle would lead to a rather complicated expression, but an approximation for large N can be obtained by imagining the poles smeared out on the semicircle, so that the sum can be evaluated as an integral. Then

$$\dot{\theta} \cong \frac{2N}{\pi}\int_0^{\pi/2}\frac{1 + \lambda^2}{\omega_0}\frac{\cos\beta \, d\beta}{1 + \lambda^4 + 2\lambda^2\cos 2\beta} = \frac{2N}{\pi\omega}\ln\left|\frac{\omega + \omega_0}{\omega - \omega_0}\right|. \quad (26)$$

This shows clearly that, for large N, $\dot{\theta}$ increases directly in proportion to N. As a sidelight, the proportionality constant, plotted in Fig. 6, is larger if the signal frequency is near the band edge. The apparent infinity for $\omega = \omega_0$ is a mathematical fiction; it resulted from smearing the poles. For any finite N, $\dot{\theta}$ is finite; thus, the curve in Fig. 6 really should

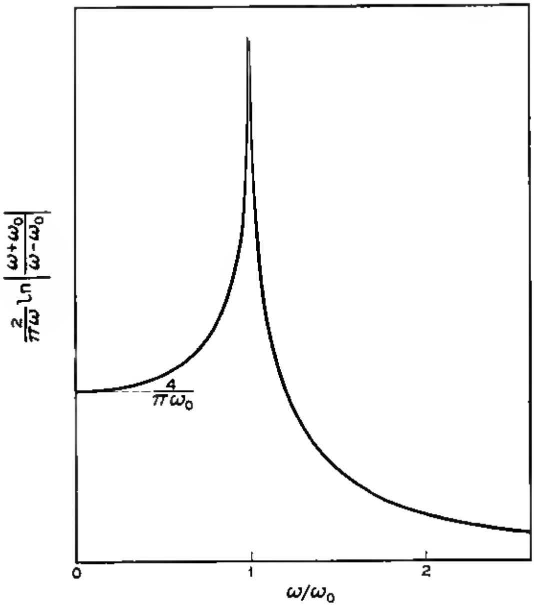

Fig. 6 — Proportionality constant.

be rounded over at the peak. For signal frequencies outside of the noise band, the detectability becomes large simply because the $1/G(\omega)$ term multiplying everything in (25) becomes large. Even straight cross-correlation would give large detectability here.

3.5 *Constant Signal*

For a constant signal, $S(t) = A$, the detectability can be written (see Appendix C)

$$d = \frac{A^2}{G(0)}\left[T - 2\sum_{k=1}^{N}\frac{1}{\gamma_k}\right]. \tag{27}$$

Note that the minus sign does not imply negative detectability; the γ's have negative real parts and so their sum will be negative. Equation (27) shows clearly that the detectability increases as the number of poles bounded away from the imaginary axis is increased.

For N-pole Butterworth noise of bandwidth ω_0, (27) becomes (exactly)

$$d = \frac{A^2}{G(0)}\left[T + \frac{2}{\omega_0 \sin(\pi/2N)}\right],$$

which for large N becomes

$$d \xrightarrow[N\to\infty]{} \frac{A^2}{G(0)}\left[T + \frac{4N}{\pi\omega_0}\right].$$

Here again the detectability grows directly in proportion to N for large N.

IV. CONCLUSIONS

We have presented solutions to some problems involving detection of the presence of known signals in gaussian noise. Thus, we are concerned with what a statistician would term hypothesis testing. Two general classes of detectors are studied, the first a maximum likelihood detector operating on a finite number of samples of the stimulus, the second an optimum integral operator treating the stimulus as a continuous function. However, the new results lie not in the general detection equations, which differ little from ones previously given, but rather in the specific solutions to these equations.

In the finite sampling case, detectability of a sinusoid or constant in bandlimited noise is computed for the cases where the samples are equally spaced time samples spread over a finite duration and where the samples are measurements of successive derivatives at one point in time. As the number of samples increases, detectability increases without bound for both cases. However, for the time samples the rate of increase is very slow for a large number of samples while for derivatives the rate becomes a linear function of the number of samples.

For optimum linear integral detection a general solution is presented for arbitrary signals in noise with a rational all-pole spectrum. The solution in closed form is sufficiently tractable so that the asymptotic behavior of certain simple signals can be evaluated as the number of poles in the noise becomes very large. The solution puts in evidence two different detection mechanisms, one involving integration of the product of the stimulus with a function derived from the signal, the other involving measurement of the derivatives of the stimulus. The first is denoted correlation detection, the second extrapolation detection. Usually, the term arising from correlation detection is the more important. However, if the stimulus is very short or if the noise spectrum has a great number of poles, the extrapolation term may become relatively large. For signals such as a sinusoid it grows without bound as the number of poles increases.

What are the implications of these solutions on previous detection results? Probably they have very little bearing on the perception prob-

lem which engendered the study, since it seems unlikely that animal sense organs embody the mechanisms implied by the solutions or that the characterization of exactly known signals in gaussian noise is appropriate. Both the solutions and the character of the stimuli differ significantly from the Fourier series bandlimited case treated by Peterson, Birdsall and Fox. In particular, the extrapolation detection does not appear in their universe. Also, we feel that the characterization of the noise as described by a correlation function is, to say the least, more suited to the present style of engineering and, to say the most, a much more satisfactory model of most detection situations.

The practical impact, if any, of the detectors developed here would seem to inhere in situations where short pieces of valuable signals must be detected and a great quantity of computing equipment is available. Such might be the case for some space communication problems.

A number of unsolved problems arise directly from the work. For a finite number of time samples of the stimulus, the optimum distribution in time of these samples is unknown. Spectra with zeros as well as poles have not been treated with anything near the elegance of the pure pole situation. Only very specific classes of signals have been studied. It would be of interest to establish which signals give unbounded and which give bounded detectability as the number of poles in the noise increases. Finally, only the case of signals known exactly has been examined. The far more difficult area involving signals with random parameters is almost untouched so far as practical solutions are concerned.

V. ACKNOWLEDGMENT

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APPENDIX A

Detection in Flat Bandlimited Noise by Estimating Derivatives

In the main body of the paper it was shown tbat, for samples which are derivatives, detectability in terms of d can be determined from (5) and (6)

$$\sum_i \beta_{ij} Z_i = S_j , \qquad j = 1, \cdots , n \tag{5}$$

$$d = \sum_j Z_j S_j . \tag{6}$$

S_j is the $j - 1$ derivative of the signal evaluated at $t = 0$ and β_{ij}, the correlation coefficient of the noise derivatives, is

$$\beta_{rs} = \begin{cases} \left(\dfrac{1}{r+s-1}\right)(-1)^{\frac{3}{2}r+\frac{1}{2}s} & \text{if } r+s \text{ is even} \\ 0 & \text{if } r+s \text{ is odd} \end{cases}$$

for flat bandlimited noise with unit rms amplitude.

Equation (5) may be written out in matrix form for odd n as

$$\begin{bmatrix} 1 & 0 & -1/3 & 0 & \cdots & 0 & \pm 1/n \\ 0 & 1/3 & 0 & -1/5 & \cdots & \mp 1/n & 0 \\ -1/3 & 0 & 1/5 & 0 & \cdots & & \\ \vdots & & & & & & \\ \pm 1/n & 0 & \cdots & & & & 1/(2n-1) \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_n \end{bmatrix}$$

and a similar form for even n.

This equation may be simplified by separating into two equations and multiplying by minus one in appropriate places to remove minus signs. Two forms occur, one for even n, the other for odd n. For n odd,

$$\begin{bmatrix} 1 & 1/3 & 1/5 & \cdots & 1/n \\ 1/3 & 1/5 & 1/7 & \cdots & \\ 1/5 & 1/7 & 1/9 & \cdots & \\ \vdots & & & & \\ 1/n & \cdots & & & 1/(2n-1) \end{bmatrix} \begin{bmatrix} Z_1 \\ -Z_3 \\ Z_5 \\ \vdots \\ \pm Z_n \end{bmatrix} = \begin{bmatrix} S_1 \\ -S_3 \\ S_5 \\ \vdots \\ \pm S_n \end{bmatrix} \tag{28}$$

and

$$\begin{bmatrix} 1/3 & 1/5 & 1/7 & \cdots & 1/n \\ 1/5 & 1/7 & 1/9 & \cdots & \\ 1/7 & 1/9 & \cdots & & \\ \vdots & & & & \\ 1/n & \cdots & & & 1/(2n-3) \end{bmatrix} \begin{bmatrix} Z_2 \\ -Z_4 \\ Z_6 \\ \vdots \\ \pm Z_{n-1} \end{bmatrix} = \begin{bmatrix} S_2 \\ -S_4 \\ S_6 \\ \vdots \\ \pm S_{n-1} \end{bmatrix}. \tag{29}$$

For n even,

$$\begin{bmatrix} 1 & 1/3 & 1/5 & \cdots & 1/(n-1) \\ 1/3 & 1/5 & 1/7 & \cdots & \\ 1/5 & 1/7 & 1/9 & \cdots & \\ \vdots & & & & \\ 1/(n-1) & & & & (2n-3) \end{bmatrix} \begin{bmatrix} Z_1 \\ -Z_3 \\ Z_5 \\ \vdots \\ \pm Z_{n-1} \end{bmatrix} = \begin{bmatrix} S_1 \\ -S_3 \\ S_5 \\ \vdots \\ \pm S_{n-1} \end{bmatrix} \tag{30}$$

and

$$\begin{bmatrix} 1/3 & 1/5 & 1/7 & \cdots & 1/(n+1) \\ 1/5 & 1/7 & 1/9 & \cdots & \\ 1/7 & 1/9 & \cdots & & \\ \vdots & & & & \\ 1/(n+1) & & & & 1/(2n-1) \end{bmatrix} \begin{bmatrix} Z_2 \\ -Z_4 \\ Z_6 \\ \vdots \\ \pm Z_n \end{bmatrix} = \begin{bmatrix} S_2 \\ -S_4 \\ S_6 \\ \vdots \\ \pm S_n \end{bmatrix}. \tag{31}$$

The determinants of these matrices can be evaluated by applying a rule attributed to Cauchy. In general, the rule says that a determinant whose ijth element is

$$M_{ij} = \frac{1}{a_i + b_j}$$

has the value

$$\left| \frac{1}{a_i + b_j} \right| = \frac{\prod_{j=1}^{n-1} \prod_{i=j+1}^{n} (a_i - a_j)(b_i - b_j)}{\prod_{j=1}^{n} \prod_{i=1}^{n} (a_i + b_j)}.$$

For the particular cases considered here, a_i and b_j have especially simple forms. For example, for (28), $a_i = 2i - 1$ and $b_j = 2j - 2$.

In addition, all cofactors of the matrices are also of Cauchy form. Hence, it is possible to invert the matrices by the method of cofactors and thus solve the equations. Such solutions are quite complex for arbitrary signals. However, an especially simple answer can be obtained for a constant since

$$S_1 = K,$$

$$S_i = 0 \qquad i \neq 1,$$

where K is the signal amplitude. Equation (6) reduces to

$$d = Z_1 S_1 = Z_1 K.$$

Z_1 may be determined by the well-known method for solving equations as the ratio of two determinants,

$$Z_1 = \frac{\begin{vmatrix} K & 1/3 & 1/5 & \cdots & 1/(2m-1) \\ 0 & 1/5 & \cdots & & \\ \vdots & & & & \\ 0 & 1/(2m+1) & & & 1/(4m-3) \end{vmatrix}}{\begin{vmatrix} 1 & 1/3 & 1/5 & \cdots & 1/(2m-1) \\ 1/3 & 1/5 & \cdots & & \\ \vdots & & & & \\ 1/(2m-1) & & & & 1/(4m-3) \end{vmatrix}},$$

where

$$m = \begin{cases} \dfrac{n+1}{2} & \text{for } n \text{ odd} \\[2ex] \dfrac{n}{2} & \text{for } n \text{ even.} \end{cases}$$

Application of Cauchy's rule and the solution for d yields

$$d = K^2 \left[\frac{(2m)!}{2^{2m-1} m!\,(m-1)!} \right]^2,$$

which is the result utilized in the main part of the paper.

APPENDIX B

In this appendix we give a general solution to the integral equation

$$\int_0^T R(t-u) Z(u)\, du = S(t) \qquad 0 \leqq t \leqq T, \tag{32}$$

where $R(t)$ is the correlation function of a noise whose spectral density is a rational function of frequency having only poles and $S(t)$ is an arbitrary known signal. The solution of the equation can be expressed in a number of different forms. The particular one developed here has

the great advantage of being an explicit function of $R(t)$ and $S(t)$ rather than involving the solution of a set of linear equations. In addition, it possesses the aesthetically pleasing property of not involving analytic continuation of $S(t)$ outside the interval $0 \leqq t \leqq T$. The noise spectral density can be written

$$G(\omega) = \left.\frac{1}{Q(p)}\right|_{p=j\omega}, \qquad Q(p) = \sum_{k=0}^{N} a_{2k}p^{2k}. \tag{33}$$

If we think of $Q(p)$ as an operator with p interpreted as d/dt we see that

$$Q(p)[R(t)] = Q(p) \int_{-\infty}^{+\infty} \frac{1}{Q(j\omega)} e^{j\omega t} \frac{d\omega}{2\pi} = \delta(t), \tag{34}$$

where $\delta(t)$ is the Dirac delta function. Operating formally on both sides of (32) with $Q(p)$ yields

$$Z_c(t) = Q(p)[S(t)] = \sum_{k=0}^{N} a_{2k}S^{(2k)}(t), \qquad 0 < t < T. \tag{35}$$

The subscript has been added to Z to indicate that this may be only part of the answer and the superscript (n) indicates n-fold differentiation with respect to time. If (32) had a $Z(t)$ solution which was continuous, then (35) would be that solution. But the fact that (35) is continuous (as it would be if $S(t)$ and its derivatives were continuous) does not prove that it is the complete solution. In fact, one can readily verify that (35) is not the complete solution by inserting it back in (32) and seeing if (32) is satisfied. It turns out that (35) is indeed part of the answer, and the remaining part is found by just this process of inserting (35) back in (32) and finding what is missing. If we imagine for the moment that $S(t)$ is extended in some arbitrary way outside the interval (so that it is Fourier transformable and the function and its derivatives go to zero at $\pm\infty$), we can write

$$\begin{aligned}\int_0^T R(t-u)Z_c(u)\,du &= \\ &\left[\int_{-\infty}^{+\infty} - \int_{-\infty}^{0} - \int_{T}^{+\infty}\right][R(t-u)Z_c(u)\,du].\end{aligned} \tag{36}$$

The first integral on the right is a normal convolution of Z_c and R, and if Z_c from (35) is substituted in we get back exactly $S(t)$. The second and third integrals are evaluated by repeated partial integration, or,

what is equivalent, by finding an exact differential expression for the integrand. We first note that $Q(p)$ can always be factored,

$$Q(p) = P(p)P(-p), \qquad P(p) = \sum_{k=0}^{N} b_k p^k, \tag{37}$$

where $P(p)$ contains only left-half plane zeros. Now define

$$U_1(t) = P(p)[S(t)] = \sum_{k=0}^{N} b_k S^{(k)}(t)$$

and (38)

$$U_2(t) = P(-p)[S(t)] = \sum_{k=0}^{N} (-1)^k b_k S^{(k)}(t).$$

The exact differential that we need is obtained by clairvoyance. It is

$$\begin{aligned} \frac{d}{du} \sum_{j=1}^{N} \sum_{i=j}^{N} & b_i U_2^{(i-j)}(u) R^{(j-1)}(t-u) \\ &= \sum_{k=0}^{N} b_k [U_2^{(k)}(u) R(t-u) - U_2(u) R^{(k)}(t-u)] \qquad (39) \\ &= Z_c(u) R(t-u) - U_2(u) \sum_{k=0}^{N} b_k R^{(k)}(t-u). \end{aligned}$$

Now, since $P(p)$ has the left-half plane zeros of $Q(p)$, the Fourier transform of $P(p)[R(t)]$ will have only right-half plane poles and thus

$$P(p)[R(t)] = \sum_{k=0}^{N} b_k R^{(k)}(t) = 0 \qquad \text{for} \quad t > 0.$$

Therefore, when we use (39) in the middle integral on the right of (36), we get

$$\int_{-\infty}^{0} R(t-u) Z_c(u)\, du = \sum_{i=0}^{N-1} \sum_{k=i}^{N-1} b_{k+1} U_2^{(k-i)}(0) R^{(i)}(t). \tag{40}$$

The third integral on the right of (36) is evaluated in a similar way, using now (39) with U_2 replaced by U_1 and b_k by $(-1)^k b_k$, and noting that $P(-p)R(t) = 0$ for $t < 0$. In this way we get

$$\int_{T}^{\infty} R(t-u) Z_c(u)\, du = \sum_{i=0}^{N-1} \sum_{k=i}^{N-1} (-1)^k b_{k+1} U_1^{(k-i)}(T) R^{(i)}(t-T). \tag{41}$$

It is interesting to note that (40) and (41) depend only on values of $S(t)$ inside the interval $0 \leqq t \leqq T$, so that the way in which $S(t)$ was

extended outside the interval does not matter. To summarize this, we find

$$\int_0^T R(t-u)Z_c(u)\,du =$$
$$S(t) - \sum_{i=0}^{N-1}\sum_{k=i}^{N-1}[b_{k+1}U_2^{(k-i)}(0)R^{(i)}(t) + (-1)^k b_{k+1}U_1^{(k-i)}(T)R^{(i)}(t-T)]. \tag{42}$$

It is now clear that, for Z_c to be the complete solution to (32), the double sum in (42) must be zero for all t in the interval. This is equivalent to the following boundary conditions on $S(t)$:

$$\left.\begin{aligned}\sum_{k=i}^{N-1} b_{k+1}U_2^{(k-i)}(0) &= 0\\ \sum_{k=i}^{N-1}(-1)^k b_{k+1}U_1^{(k-i)}(T) &= 0\end{aligned}\right\}\quad i = 0, 1, \cdots, (N-1). \tag{43}$$

If the signal is such that these conditions are not satisfied, then (32) has a solution only if $Z(t)$ includes delta functions and their derivatives, that is

$$Z(t) = Z_c(t) + \sum_{i=0}^{N-1}[\alpha_i\delta^{(i)}(t) + \beta_i\delta^{(i)}(t-T)]. \tag{44}$$

If this is used in (32), the delta functions bring out R and its derivatives evaluated at t and $t - T$, and the α's and β's can be directly identified as

$$\begin{aligned}\alpha_i &= \sum_{k=i}^{N-1} b_{k+1}U_2^{(k-i)}(0),\\ \beta_i &= \sum_{k=i}^{N-1}(-1)^k b_{k+1}U_1^{(k-i)}(T).\end{aligned} \tag{45}$$

The detectability for a Z which satisfies (32) is

$$d = \int_0^T S(t)Z(t)\,dt;$$

thus, using (44),

$$d = \int_0^T S(t)Z_c(t)\,dt + \sum_{i=0}^{N-1}(-1)^i[\alpha_i S^{(i)}(0) + \beta_i S^{(i)}(T)]. \tag{46}$$

This can be put in a slightly different form which may be more convenient by again partially integrating. Using another exact differential obtained by clairvoyance, which is

$$\frac{d}{dt}\left[\sum_{=0}^{N-1}\sum_{k=i}^{N-1}(-1)^{k+i}b_{k+1}U_1^{(k-i)}(t)S^{(i)}(t)\right] = -S(t)Z_c(t) + U_1^2(t) \quad (47)$$

and observing that when this is inserted in (46) the terms evaluated at T cancel, we get

$$\begin{aligned} d = &\int_0^T U_1^2(t)\,dt \\ &+ \sum_{i=0}^{N-1}(-1)^i S^{(i)}(0)\sum_{k=i}^{N-1} b_{k+1}[U_2^{(k-i)}(0) + (-1)^k U_1^{(k-i)}(0)] \end{aligned} \quad (48)$$

or, equivalently, in an operator notation,

$$\begin{aligned} d = &\int_0^T U_1^2(t)\,dt \\ &+ \sum_{i=0}^{N-1} S^{(i)}(0)[(-1)^i P_i(p)U_2(t) + P_i(-p)U_1(t)]_{t=0}, \end{aligned} \quad (49)$$

where

$$P_i(x) = \sum_{k=i}^{N-1} b_{k+1}x^{k-i}.$$

In this form the summation only involves derivatives at $t = 0$, which in some cases simplifies the algebra of a solution.

APPENDIX C

As a particular example, we calculate the detectability d for the case in which the signal is an exponentially damped sine wave,

$$S(t) = Ae^{-\alpha t}\sin(\omega t + \Phi) = \bar{A}e^{\lambda t} + \bar{A}^*e^{\lambda^* t}, \quad (50)$$

where

$$\bar{A} = \frac{A}{2j}e^{j\Phi} \qquad \text{and} \qquad \lambda = -\alpha + j\omega$$

and the asterisk denotes complex conjugate. Using this in the expression

for detectability (20) or (49), we find that the second term—call it d_d — becomes

$$d_d = \sum_{i=0}^{N-1} S^{(i)}(0)\{[(-1)^i P_i(p)P(-p) + P_i(-p)P(p)]S(t)\}_{t=0}$$
$$= 2 \text{ Re} \sum_{i=0}^{N-1} \{\bar{A}^2[(-\lambda)^i P_i(\lambda)P(-\lambda) + \lambda^i P_i(-\lambda)P(\lambda)] \tag{51}$$
$$+ |\bar{A}|^2[(-\lambda^*)^i P_i(\lambda)P(-\lambda) + \lambda^{*i} P_i(-\lambda)P(\lambda)]\}.$$

The notation Re means "real part of." From the definition of $P_i(x)$, one can readily verify that

$$\sum_{i=0}^{N-1} y^i P_i(x) = \frac{P(x) - P(y)}{x - y}, \tag{52}$$

and this allows (51) to be greatly simplified:

$$d_d = 2 \text{ Re}\left[\bar{A}^2 \frac{P^2(\lambda) - P^2(-\lambda)}{2\lambda}\right]$$
$$+ 2|\bar{A}|^2 \frac{|P(\lambda)|^2 - |P(-\lambda)|^2}{\lambda + \lambda^*}. \tag{53}$$

The first term in the detectability, (20) or (49) is simply an integral,

$$\int_0^T U_1^2(t)\,dt = \int_0^T [\bar{A}P(\lambda)e^{\lambda t} + \bar{A}^*P(\lambda^*)e^{\lambda^* t}]^2\,dt$$
$$= 2 \text{ Re}\left[\bar{A}^2 P^2(\lambda)\frac{e^{2\lambda T} - 1}{2\lambda}\right] + 2|AP(\lambda)|^2 \frac{e^{(\lambda+\lambda^*)T} - 1}{\lambda + \lambda^*}. \tag{54}$$

Combining (53) and (54), we get

$$d = 2 \text{ Re}\left[\bar{A}^2 \frac{P^2(\lambda)e^{2\lambda T} - P^2(-\lambda)}{2\lambda}\right]$$
$$+ 2|\bar{A}|^2\left[\frac{|P(\lambda)|^2 e^{(\lambda+\lambda^*)T} - |P(-\lambda)|^2}{\lambda + \lambda^*}\right] \tag{55}$$

which is the general solution for any damped sinusoid.

Three special cases are now considered, the pure exponential, the pure sine wave, and a constant (DC) signal. For a pure exponential signal, $\lambda \to -\alpha$ where α is real in (55), giving

$$d = \frac{A^2}{2\alpha}[P^2(\alpha) - P^2(-\alpha)e^{-2\alpha T}]. \tag{56}$$

For a pure sine wave signal, $\lambda \to j\omega$. The second term in (55) requires a little special treatment, but it is easily shown that

$$\frac{|P(\lambda)|^2 e^{(\lambda+\lambda^*)T} - |P(-\lambda)|^2}{\lambda + \lambda^*} \xrightarrow[\alpha \to 0]{}$$

$$T\,|P(j\omega)|^2 + \frac{P(-j\omega)}{j}\frac{dP(j\omega)}{d\omega} - \frac{P(j\omega)}{j}\frac{dP(-j\omega)}{d\omega}.$$

Now, $P(j\omega)$ is simply the reciprocal of the transfer function of the noise filter at the frequency ω; that is,

$$P(j\omega) = 1/H(j\omega) = \frac{e^{j\theta(\omega)}}{\sqrt{G(\omega)}},$$

where $\theta(\omega)$ is the phase lag of the noise filter. Using this expression,

$$d = \frac{A^2}{2G(\omega)}\left[T + 2\dot{\theta}(\omega) - \frac{\sin 2(\omega T + \theta + \Phi) + \sin 2(\theta - \Phi)}{2\omega}\right], \quad (57)$$

where $\dot{\theta} = d\theta/d\omega$.

For a constant signal we can simply take (56) and let $\alpha \to 0$, which gives

$$\begin{aligned} d &= A^2\left[TP^2(0) + \left.\frac{dP^2(\alpha)}{d\alpha}\right|_{\alpha=0}\right] \\ &= \frac{A^2}{G(0)}\left[T + 2\frac{b_1}{b_0}\right] = \frac{A^2}{G(0)}\left[T - 2\sum_{k=1}^{N} 1/\gamma_k\right]. \end{aligned} \quad (58)$$

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G. E. Schindler, Jr., New Editor of B.S.T.J.

G. E. Schindler, Jr., was appointed editor of the Bell System Technical Journal, effective January 1, 1961. Mr. Schindler studied chemical engineering at the Carnegie Institute of Technology, received the bachelor of science degree from the University of Chicago, and received the master of arts degree in English literature and languages from the University of Pittsburgh. After additional graduate work at the University of Chicago, Mr. Schindler joined Bell Telephone Laboratories in 1953. He was editor of the Bell Laboratories Record from 1957 to 1959, and most recently was with the Public Relations department of the A. T. & T. Co.